Proof that the Harmonic Series Diverges, by Induction

we need to show that

$$\sum_{i=1}^{k_n} \frac{1}{i} \geq n$$

basis step: $\sum_{i=1}^{k_1} \frac{1}{i} \geq 1$

$\frac{1}{1} = 1 \quad 1 \geq 1 \quad \checkmark$

inductive hypothesis:

$$\sum_{i=1}^{k_n} \frac{1}{i} \geq n$$

inductive step: $\sum_{i=1}^{k_{n+1}} \frac{1}{i} \overset{?}{\geq} n+1$

$$\sum_{i=1}^{k_{n+1}} \frac{1}{i} = \sum_{i=1}^{4k_n} \frac{1}{i} = \sum_{i=1}^{k_n} \frac{1}{i} + \sum_{i=k_n+1}^{4k_n} \frac{1}{i}$$

using the I. H.: $= n + \sum_{i=k_n+1}^{4k_n} \frac{1}{i} \overset{?}{\geq} n+1$

to finish the proof, we need to show that $\sum_{i=k_n+1}^{4k_n} \frac{1}{i} \geq 1$

since $\sum_{i=1}^{k_{n+1}} \frac{1}{i} = \sum_{i=1}^{k_n} \frac{1}{i} + \sum_{i=k_n+1}^{4k_n} \frac{1}{i}$

and $\sum_{i=k_n+1}^{4k_n} \frac{1}{i} \geq 1$, $\boxed{\sum_{i=1}^{k_{n+1}} \frac{1}{i} \geq n+1}$

k_n is the number of elements needed to sum to at least n

$k_1 = 1 \quad k_2 = 4 \quad k_3 = 11 \quad k_4 = 63$

since it's to <u>at least</u> n, we can say:

$k_3 = 16 \quad k_4 = 64$

the pattern goes $\boxed{k_{n+1} = 4k_n}$

since we have an inequality, if we can show $x > y$ and $y > z$, then $x > z$. so, let's add the smallest value of $\frac{1}{i}$ each time instead of each value of $\frac{1}{i}$

$$\sum_{i=k_n+1}^{4k_n} \frac{1}{4k_n} = 3\cancel{k_n}\left(\frac{1}{4\cancel{k_n}}\right) = \frac{3}{4}$$

(k_n+1 to $4k_n$ is $3k_n$ terms)

not quite good enough. let's break up the summation again and do the same.

$$\sum_{i=k_n+1}^{4k_n} \frac{1}{i} = \sum_{i=k_n+1}^{2k_n} \frac{1}{i} + \sum_{i=2k_n+1}^{4k_n} \frac{1}{i}$$

$$> \sum_{i=k_n+1}^{2k_n} \frac{1}{2k_n} + \sum_{i=2k_n+1}^{4k_n} \frac{1}{4k_n} = \cancel{k_n}\left(\frac{1}{2\cancel{k_n}}\right) + 2\cancel{k_n}\left(\frac{1}{4\cancel{k_n}}\right)$$

$$= \frac{1}{2} + \frac{2}{4} = 1$$